CENTRAL FORCE

If a force acts on a particle in such a way that it is always directed towards or away from a fixed centre and its magnitude depends only upon the distance (r) from the centre, then this force is called central force.

A central force is represented by,

$$F = f(r)\hat{r} = f(r)\frac{\vec{r}}{r}$$

Where, $f(r)$ is a function of distance r only

and $\hat{r}$ is a unit vector along r from fixed centre.

The force is attractive or repulsive if $f(r) < 0$ or $f(r) > 0$ respectively.

A central force is always a conservative force and if $V(r)$ is the potential energy, then

$$f(r) = -\frac{\partial V}{\partial r} \text{ or } F = -\frac{\partial V}{\partial r}\frac{\vec{r}}{r}$$

The potential energy for central force ~~and if $V(r)$ is the potential energy, then f~~ depends only on the distance r and hence the system possesses spherical symmetry.

Features of central force motion

(1) Angular Momentum is conserved:

If a particle is under central force then it is expressed as $\vec{F}(r) = f(r)\hat{r}$, where $\hat{r} = \vec{r}/r$ is a unit vector along the direction of $\vec{r}$.

The torque acting on the particle is given by,

$$\vec{\tau} = (\vec{r} \times \vec{F}) = \vec{r} \times f(r)\hat{r}$$

$$= \vec{r} \times f(r)\hat{r}\frac{\vec{r}}{r}$$

$$= 0 \qquad [\vec{r} \times \vec{r} = 0]$$

Scanned by CamScanner

So no external torque acts on the particle.

But torque is the rate of change of angular momentum $\vec{L}$

$\vec{\tau} = \frac{d\vec{L}}{dt}$, where $\vec{L}$ = Angular momentum.

So, if $\vec{\tau} = 0$ then $\frac{d\vec{L}}{dt} = 0 \Rightarrow \boxed{\vec{L} = \text{const.}}$

Hence when a particle moves under a central force then the angular momentum w.r.t centre of force is a const. of motion.

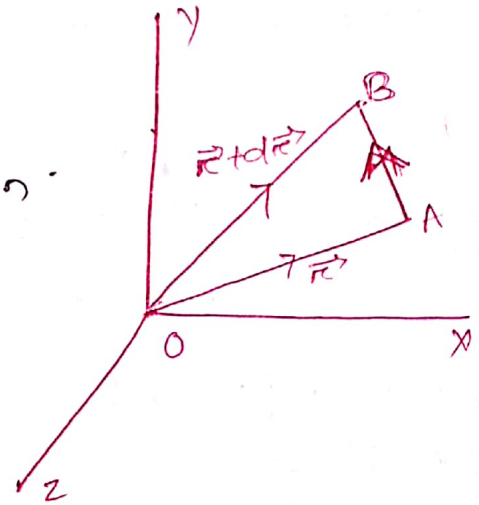
2) Motion under central force is confined in a plane :-

We know $\vec{L} = \vec{r} \times \vec{p}$. Also $\vec{L}$ is const. in magnitude and direction as $\vec{L} = \text{const.}$ for central force motion. From the rule of cross product we know that $\vec{r}$ is perpendicular to $\vec{L}$ and $\vec{L}$ is fixed in the direction of motion.

As a result $\vec{r}$ can move in a plane perpendicular to $\vec{L}$. This means that the motion of the particle under central force takes place in a plane.

3) Areal velocity of a particle moving under central force remains const. :-

Let $\vec{r}$ and $\vec{r} + d\vec{r}$ be position vector of a particle at point A and B w.r.t origin of coordinate system.



$\vec{AB} = \vec{r} + d\vec{r} - \vec{r}$

$= d\vec{r}$ $\quad [\because \vec{AB} = \vec{OB} - \vec{OA}]$

The area covered in time dt from A to B is, $d\vec{A} = \frac{1}{2} \vec{r} \times d\vec{r}$

$\Rightarrow \frac{d\vec{A}}{dt} = \frac{1}{2} \vec{r} \times \frac{d\vec{r}}{dt}$

$\Rightarrow$ Areal velocity $= \frac{1}{2} \vec{r} \times \frac{m\, d\vec{r}/dt}{m}$

Scanned by CamScanner

turn it $\Rightarrow$ Areal velocity $= \frac{1}{2m}(\vec{r} \times \vec{p}) = \frac{\vec{L}}{2m}$ = constant.

as $\vec{L}$ = const. for a particles under central force motion.

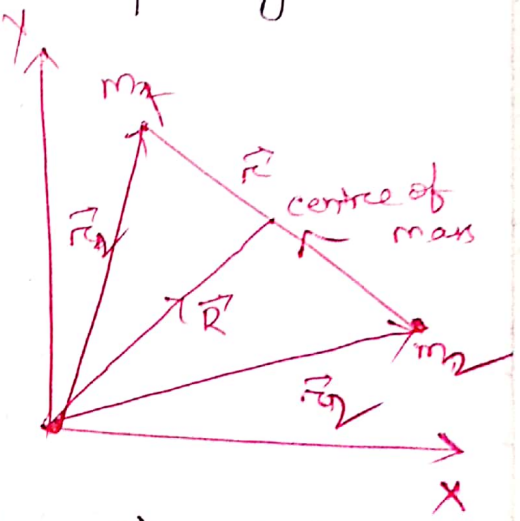
Hence we conclude that the areal velocity of a particle ler central force remains const. w.r.t the origin of the rdinate system.

the motion.

Two Body Problem and Its reduction to One-body problem

In classical mechanics, the two-body problem is to determine the motion of 2 point particles that interact only with each other. Common examples include a satellite orbiting a planet, a planet orbiting a star (planet-sun system), hydrogen atom (consisting of an e^- revolving around a proton), positronium, any diatomic molecule like H_2, HCl etc.

The two-body can be re-formulated as two one-body problems, a trivial one and one that involves solving for the motion of one particle in an external potential. Since many one-body-problems can be solved, exactly the corresponding two-body problem can be solved.



Lets consider an isolated system of two particles under a central force $\vec{f}$ which depends on the distance b/w 2 particles and is directed along the line joining the 2 particles. If m_1 and m_2 be the masses of the two particles under a central forces having position vector $\vec{r}_1$ and $\vec{r}_2$ w.r.t origin as shown on figure then vector distance of m_1 from m_2 is, $\vec{r} = \vec{r}_1 - \vec{r}_2$

$\therefore |\vec{r}| = r = |\vec{r}_1 - \vec{r}_2|$

Scanned by CamScanner

The 2 particles exert central force on each other the line joining m_1 and m_2 i.e. along r. If $\vec{F}_{12}$ is force exerted by m_2 on m_1 then $\vec{F}_{21}$ is force exerted on m_2 by m_1 i.e. action and reaction pair given by,

$$\vec{F}_{12} = -\vec{F}_{21} \quad \text{(From Newton's 3rd law)}$$

So if we write, $\vec{F}_{12} = f(r)\,\hat{r}$ then $\vec{F}_{21} = -f(r)\,\hat{r}$

Then the equations of motion for m_1 & m_2 are given by,

$$m_1 \frac{d^2 r_1}{dt^2} = f(r)\,\hat{r}$$

$$\Rightarrow m_1 \ddot{\vec{r}}_1 = f(r)\,\hat{r} \qquad \text{---------- (1)}$$

and $$m_2 \frac{d^2 r_2}{dt^2} = -f(r)\,\hat{r}$$

$$\Rightarrow m_2 \ddot{\vec{r}}_2 = -f(r)\,\hat{r} \qquad \text{---------- (2)}$$

From (1) and (2) we get,

$$\ddot{\vec{r}}_1 = \frac{f(r)\,\hat{r}}{m_1} \quad \text{and} \quad \ddot{\vec{r}}_2 = \frac{-f(r)\,\hat{r}}{m_2}$$

Then subtracting,

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = f(r)\,\hat{r}\left(\tfrac{1}{m_1} + \tfrac{1}{m_2}\right)$$

$$\Rightarrow \frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = f(r)\,\hat{r}\left(\frac{m_1 + m_2}{m_1 m_2}\right)$$

$$\Rightarrow \left(\frac{m_1 m_2}{m_1 + m_2}\right)\frac{d^2 \vec{r}}{dt^2} = f(r)\,\hat{r}$$

$$\Rightarrow \mu \ddot{\vec{r}} = f(r)\,\hat{r} \qquad \text{---------- (3)}$$

Where we have used, $\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$ & known as the reduced mass of the system.

Scanned by CamScanner

But if we consider the centre of mass co-ordinate system or centre of mass frame then $\vec{R} = 0$.

So the centre of mass coordinates of m_1 and m_2 are $\frac{\mu}{m_1}\vec{r}$ and $-\frac{\mu}{m_2}\vec{r}$ respectively.

The Energy equation and Energy Diagram

The equation of motion under central force using the concept of reduced mass is given by,

$$\mu\ddot{\vec{r}} = f(r)\hat{r} \qquad \text{---------- (1)}$$

where μ is reduced mass.

$f(r)$ is magnitude of central force.

So, if we introduce the plane polar co-ordinates r and θ in the plane of motion then accelaration of the particle has two components $\vec{a_r}$ - the radial accelaration given by $(\ddot{r} - r\dot{\theta}^2)\hat{r}$ and transverse accelaration $\vec{a_\theta}$ given by $(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$.

The polar form of eq^n (1) becomes,

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad \text{---------- (2)}$$

and $$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad \text{---------- (3)}$$

As central force has no tangential component as it is a function of r only and independent of θ. The angular momentum of particle remains conserved

i.e. $L = \mu r^2\omega = \text{const.}$ $\quad [\because L = r \times p = r \times \mu r\omega]$

i.e. $\boxed{L = \mu r^2\dot{\theta}}$ is constant ---------- (4)

Scanned by CamScanner

Eqn (3) is similar to the equation of motion for a particle of mass μ acted on by a force $f(r)\hat{r}$. So the two problem has been reduced to one body problem. In other words the problem of calculating the two position vectors $\vec{r_1}$ and $\vec{r_2}$ of two particles of mass m_1 and m_2 have been reduced to finding a single vector $\vec{r}$. If we are able to solve equation (3) for $\vec{r}$ then we can easily obtain the values of $\vec{r_1}$ and $\vec{r_2}$ in terms of centre of mass coordinates.

The position vector of the centre of mass of 2 body system is given by,

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1+m_2} \quad \text{---- (5)}$$

Also from $\vec{r} = \vec{r_1} - \vec{r_2}$

$$\Rightarrow \vec{r_2} = \vec{r_1} - \vec{r}$$

Hence, $\vec{R} = \frac{m_1\vec{r_1} + m_2(\vec{r_1} - \vec{r})}{m_1+m_2}$

$$\Rightarrow \vec{R} = \frac{(m_1+m_2)\vec{r_1} - m_2\vec{r}}{(m_1+m_2)}$$

$$\Rightarrow \vec{R} = \vec{r_1} - \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$$

$$\Rightarrow \vec{r_1} = \vec{R} + \left(\frac{m_2\vec{r}}{m_1+m_2}\right)$$

Also $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass of system.

$$\Rightarrow \vec{r_1} = \vec{R} + \frac{\mu}{m_1}\left(\frac{m_1m_2}{m_1+m_2}\right)\vec{r} \quad \text{---- (6)}$$

$$= \vec{R} + \frac{\mu}{m_1}\vec{r}$$

and $\vec{r_2} = \vec{R} - \left(\frac{m_1}{m_1+m_2}\right)\vec{r} \Rightarrow \vec{r_2} = \vec{R} - \frac{\mu}{m_2}\vec{r}$ ---- (7)

Scanned by CamScanner

From eqn (2) we get,

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = f(r)$$

$$\Rightarrow \mu\ddot{r} - \mu r\left(\frac{L}{\mu r^2}\right)^2 = f(r)$$

$$\Rightarrow \mu\ddot{r} - \frac{L^2}{\mu r^3} = f(r) \qquad \text{——— (8)}$$

But central force is conservative and hence can be expressed in term of potential energy U as $f(r) = -\frac{dU}{dr}$.

$\therefore$ Eqn (8) becomes $\mu\ddot{r} - \frac{L^2}{\mu r^3} = -\frac{dU}{dr}$

$$\Rightarrow \mu\ddot{r} = -\frac{dU}{dr} + \frac{L^2}{\mu r^3}$$

$$\Rightarrow \mu\ddot{r} = -\frac{d}{dr}\left[U + \frac{L^2}{2\mu r^2}\right]$$

$$\Rightarrow \mu\ddot{r}\dot{r} = -\frac{d}{dr}\left[U + \frac{L^2}{2\mu r^2}\right]\dot{r}$$

$$\Rightarrow \frac{d}{dt}\left[\tfrac{1}{2}\mu\dot{r}^2\right] = -\frac{d}{dt}\left[U + \frac{L^2}{2\mu r^2}\right]$$

$$\Rightarrow \frac{d}{dt}\left[\tfrac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + U\right] = 0$$

$$\Rightarrow \tfrac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + U = \text{const.} \qquad \text{——— (9)}$$

Using $L = \mu r^2\dot{\theta}$ in above eqn we get,

$$\frac{1}{2}\mu\dot{r}^2 + \frac{(\mu r^2\dot{\theta})^2}{2\mu r^2} + U = \text{const.}$$

$$\Rightarrow \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\theta}^2 + U = \text{const.}$$

$$\Rightarrow \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U = \text{const.} \qquad \text{——— (10)}$$

Scanned by CamScanner

But $\frac{1}{2}\mu[\dot{r}^2 + r^2\dot{\theta}^2] = \frac{1}{2}\mu v^2$:

is K.E of μ as $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ in polar coordinate

So eqn (10) becomes,

$$K.E + U = \text{const.} \quad \text{——— (11)}$$

which clearly shows that the left hand side of above equation is the total energy of the particle under centre force. So we conclude that for a particle under central force motion, the total energy remains constant.

Hence we write $E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U = \text{const.}$ — (12)

Eqn (12) is written as, $E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + U(r)$ ——— (13)

which looks like the energy equation for a particle moving in one dimension. But if we introduce,

$$\frac{L^2}{2\mu r^2} + U(r) = U_{eff}(r) \quad \text{— (14)}$$

Then eqn (13) becomes,

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) \quad \text{——— (15)}$$

Where $U_{eff}(r)$ is called effective potential energy or potential which differs from true potential energy by $L^2/2\mu r^2$ known as the centrifugal potential corresponding to the centrifugal force which is a pseudo force. Equations (13) and (15) represent energy equations in central force motion but the eqn (15) is easy to handle with as it contains a single coordinate r in contrast to eqn (13), which contains 2 co-ordinate r & θ. In other words eqn (15) is similar to the eqn for the energy of a particle mass μ constrained to move along a straight line with kinetic energy $\frac{1}{2}\mu\dot{r}^2$ and potential energy $U_{eff}(r)$.

Scanned by CamScanner

Eqn (12) involves only the radial motion. This helps us to use the Energy diagram to know about the features of radial motion.

From attractive gravitational force or inverse square law of force we know,

$$f(r) = \frac{-Gm_1m_2}{r^2} = \frac{-k}{r^2}$$

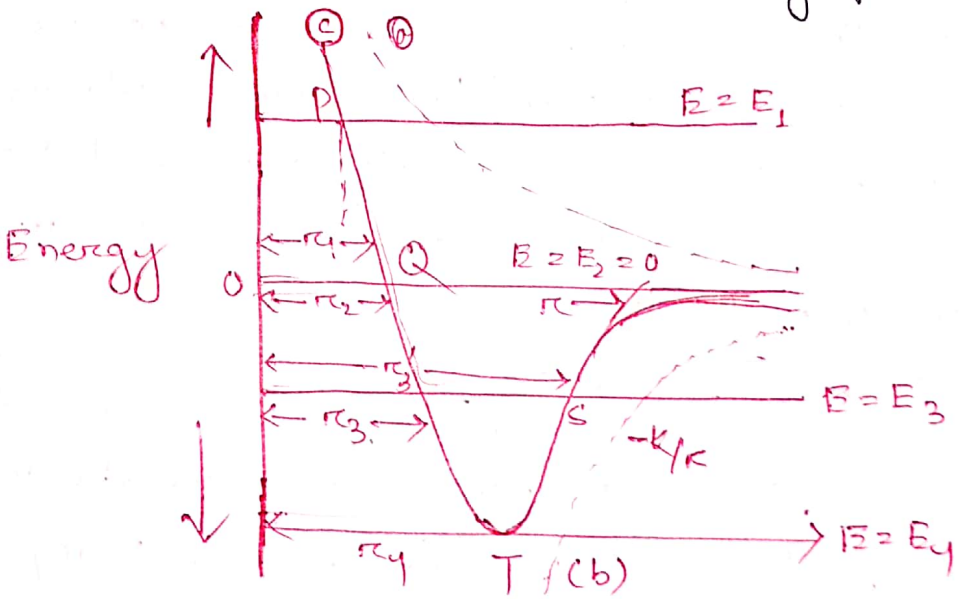
where, $k = Gm_1m_2$ is a const.. But from relation,

$$U = -\int f(r)\,dr = -\int \frac{k}{r^2}\,dr$$

$$\Rightarrow U = -k/r$$

So that $U_{eff} = \frac{L^2}{2\mu r^2} + U(r)$ becomes $U_{eff} = \frac{L^2}{2\mu r^2} - \frac{k}{r}$.

The variation of $\frac{L^2}{2\mu r^2}$ with r is shown in graph (a) and variation of $-k/r$ with r is shown in graph (b). The variation of effective potential with distance r is shown in graph (c).



The energy diagram showing different values of energy which determine nature of motion as discussed below.

Case-1 :- Total energy $E = E_1 > 0$

The radial K.E is given by $KE = E - U_{eff}$

$\Rightarrow \frac{1}{2}\mu\dot{r}^2 = E - U_{eff}$ which is always positive.

Scanned by CamScanner

The point where radial KE vanishes is called turning point. So, at point P, we have $E = U_{eff} \Rightarrow KE = 0$. As a result r_1 the minimum turning point for particle with energy E_1. But r can not be less than r_1 as in this case $U_{eff} > E$ then $E_1 - U_{eff}$ will be negative which is not possible as $KE > 0$. The particle will come from ∞ towards other then incoming particle can not come closer than r_1 and then returns back to infinity. The motion is unbounded and path is Hyperbola.

Case-2 :- Total energy $E = E_2 = 0$

For total energy $E = E_2 = 0$, there is only one turning point at Q where $OQ = r_2$. So in this case $r = r_2 > r_1$ and this case is similar to case 1. The motion of particle is unbounded but the path is a parabola.

Case-3 :- Total energy $E = E_3 < 0$

In this case the energy line cuts the curve for U_{eff} at 2 points R and S for which the radial distances are r_3 and r_3'. Hence in this case are two turning points and the motion is bounded between r_3 and r_3' also known a apsidal distances. The path is an Ellipse.

Case-4 :- Total energy $E = E_4 = U_{eff}'$

In this case the Energy line meets the curve for U_{eff} at point T for which the radial distance is r_4 for which U_{eff} has minimum value. So at this point T of the curve $\frac{dU_{eff}}{dr} = 0$.

$$\Rightarrow \frac{d}{dr}\left[\frac{L^2}{2\mu r^2} - \frac{K}{r}\right] = 0$$

$$\Rightarrow \frac{-L^2}{2\mu r^3} + \frac{K}{r^2} = 0 \Rightarrow \frac{L^2}{2\mu r^3} = \frac{K}{r^2}$$

$$\Rightarrow \boxed{r = \frac{L^2}{2\mu K}}$$

Scanned by CamScanner

The minimum value of total energy is given by,

$$E_{min} = (U_{eff})_{min}$$

$$= \frac{L^2}{2\mu r^2} - \frac{K}{r}$$

$$= \frac{L^2}{2\mu\left(\frac{L^2}{\mu K}\right)^2} - \frac{K}{L^2/\mu K}$$

$$\Rightarrow \boxed{E_{min} = -\frac{\mu K^2}{2L^2}}$$

Hence for $E = E_0 = U_{eff}$ the motion occurs only at one value of radial distance $r = r_0 = \frac{L^2}{\mu K}$. The path of the particle is circle of radius $L^2/\mu K$ having total energy negative.

* Tangent acc^n :— Tangent to the path of the particle's motion (if the speed is not const^t).

* Radial acc^n :— centripetal acc^n is along the radius.

* Lagrangian mechanics :— the trajectory of a system of particles is derived by solving the Lagrange eq^n.

Scanned by CamScanner